15

COLLIDING ELECTROMAGNETIC WAVES: THE BELL–SZEKERES SOLUTION

In an important paper, Bell and Szekeres (1974) gave an exact solution which describes the collision and subsequent interaction of two electromagnetic plane waves. This solution, which appears to be remarkably simple, will be described in this chapter. Other exact solutions describing colliding electromagnetic waves will be described in Chapters 16 and 17, after more powerful techniques have been developed.

15.1 The Bell–Szekeres solution

Bell and Szekeres have considered a very simple situation involving a collision of two step electromagnetic waves. Prior to the collision, the waves may be described by the familiar line element (4.15). In regions I and II, this takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2+Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{15.1}$$

and the step wave is given by $\Phi_{22} = a^2\Theta(u)$. The opposing wave with $\Phi_{00} = b^2\Theta(v)$ may be described in an identical way with the null coordinate u replaced by v, and a different space-like coordinate replacing r.

The two electromagnetic field components in the initial regions I, II and III are taken to be

$$\Phi_2 = a\Theta(u), \qquad \text{and} \qquad \Phi_0 = b\Theta(v). \tag{15.2}$$

The fact that both of these expressions are real simultaneously indicates that the polarization of the two electromagnetic waves is aligned. In addition, they have identical step profiles. The relative amplitudes a and b may, of course, both be equated to unity when convenient. The components of the Weyl tensor are initially all zero.

By this stage, we are familiar with the fact that it is appropriate to transform the metrics in all regions into Rosen form. Accordingly, the line elements in the initial regions may be taken in the forms of (4.19) and (4.20), namely

$$\begin{array}{lll} \text{I} \;\; (u<0, v<0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 & \\ \text{II} \;\; (u\ge 0, v<0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au(\mathrm{d}x^2+\mathrm{d}y^2) & (15.3) \\ \text{III} \, (u<0, v\ge 0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 bv(\mathrm{d}x^2+\mathrm{d}y^2). & \end{array}$$

The initial-value problem is now well set, and it remains to find the unique solution in the interaction region.

The field equations for colliding electromagnetic waves have already been obtained in Chapter 6. In terms of the metric functions of the Szekeres line element (6.20), the gravitational field equations for the interaction region take the form (6.22), and Maxwell's equations are given by (6.21). The appropriate boundary conditions for this situation have been described in Chapter 7.

As usual, it may be noted that (6.22a) can immediately be integrated to give

$$e^{-U} = f(u) + g(v) \tag{15.4}$$

where f and g are arbitrary decreasing functions in the interaction region. From the junction conditions it can be seen that, in this case, these functions necessarily take the forms:

$$f = \tfrac{1}{2} - \sin^2 au, \qquad g = \tfrac{1}{2} - \sin^2 bv. \tag{15.5}$$

The solution of the complete set of equations (6.21,22) given by Bell and Szekeres is

$$\begin{aligned} U &= -\log\cos(au - bv) - \log\cos(au + bv) \\ V &= \log\cos(au - bv) - \log\cos(au + bv) \\ W &= 0, \qquad M = 0, \qquad \Phi_2 = a, \qquad \Phi_0 = b. \end{aligned} \tag{15.6}$$

It can be seen that these functions satisfy the required O'Brien–Synge boundary conditions. Thus the line element in the interaction region IV is

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2(au - bv)\mathrm{d}x^2 - \cos^2(au + bv)\mathrm{d}y^2. \tag{15.7}$$

This line element is in fact one form of the Bertotti–Robinson solution (Bertotti 1959, Robinson 1959), which is known to be conformally flat. The coordinate transformation

$$\begin{aligned} t + r &= \coth\left(\frac{1}{2}\mathrm{sech}^{-1}\cos(au + bv) - \frac{y}{2q}\right) \\ t - r &= -\tanh\left(\frac{1}{2}\mathrm{sech}^{-1}\cos(au + bv) + \frac{y}{2q}\right) \\ \theta &= au - bv + \frac{\pi}{2}, \qquad\qquad \phi = \frac{x}{2} \end{aligned} \tag{15.8}$$

transforms the line element (15.7) into the more familiar form

$$\mathrm{d}s^2 = \frac{q^2}{r^2}(\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2). \tag{15.9}$$

Halilsoy (1987) has considered an application of the transformation $u \to u'(u)$, $v \to v'(v)$ to the above solution. However, this does not lead to any new results that are of physical significance.

15.2 The structure of the solution

The above solution describes the collision of two step electromagnetic waves whose polarization vectors are aligned. This follows from the fact that Φ_{02} is real, and $W = 0$.

As has been pointed out above, the solution inside the interaction region IV is conformally flat. However, as Bell and Szekeres have shown, there are necessarily discontinuities in the derivatives of the metric function V across the initial boundaries of this region. These manifest themselves as the impulsive gravitational waves

$$\Psi_4 = -a \tan bv \, \delta(u)\Theta(v), \qquad \Psi_0 = -b \tan au \, \delta(v)\Theta(u) \tag{15.10}$$

which may be considered to be generated by the collision.

It is in fact a general feature of colliding electromagnetic plane waves that gravitational waves are always generated by the collision. This can easily be demonstrated by considering the field equations (6.22*d,e*). In the interaction region, Φ_2 and Φ_0 are necessarily both non-zero. It follows that V and W can not both remain constant. Thus, the congruences tangent to both waves must begin to shear, and it can be seen from (6.23) that the components of the Weyl tensor Ψ_0 and Ψ_4, and possibly Ψ_2, will necessarily appear. In this case, the discontinuity in the electromagnetic components causes impulsive gravitational waves to be generated. It can also be shown that smooth-fronted electromagnetic waves would generate smooth-fronted or step gravitational waves that wouls persist through the interaction region.

It may also be observed from (15.10) that difficulties occur at the points $u = \pi/2a$, $v = 0$ and $v = \pi/2b$, $u = 0$. At first sight, it would appear that curvature singularities occur at these points (Matzner and Tipler, 1984). However, curvature tensors of this type can be interpreted as distributions (see Geroch and Traschen, 1987) and, in this case, the standard definition of a curvature singularity in terms of unboundedness in a parallely propagated frame is inapplicable.

The global structure of this solution has been analysed in detail by Clarke and Hayward (1989). They have confirmed that, in regions II and III, the surfaces on which $f = -1/2$ and $g = -1/2$ respectively behave as 'fold' singularities similar to those of the Khan–Penrose solution described in Section 8.2.

Initially, the most surprising feature of the Bell–Szekeres solution was that a space-like curvature singularity does not occur in region IV. In this case, the surface on which $f+g=0$, which is here given by $au+bv=\pi/2$, turns out to be merely a coordinate singularity since the curvature tensor is clearly bounded. Bell and Szekeres have demonstrated how this singularity can be removed by considering the following coordinate transformation:

$$\begin{aligned}\tilde T &= -\cos(au+bv)\cosh cy\\ \tilde Z &= \cos(au+bv)\sinh cy\\ \tilde X &= \cos(au-bv)\cos cx\\ \tilde Y &= \cos(au-bv)\sin cx\end{aligned} \tag{15.11}$$

where $c=\sqrt{2ab}$. With this, the line element (15.6) becomes

$$\begin{aligned}\mathrm{d}s^2 = &\left(\frac{(1+\tilde Z^2)\mathrm{d}\tilde T^2 - 2\tilde T\tilde Z\mathrm{d}\tilde T\mathrm{d}\tilde Z - (1-\tilde T^2)\mathrm{d}\tilde Z^2}{c^2(1-\tilde T^2+\tilde Z^2)}\right)\\ &-\left(\frac{(1-\tilde Y^2)\mathrm{d}\tilde X^2 + 2\tilde X\tilde Y\mathrm{d}\tilde X\mathrm{d}\tilde Y + (1-\tilde X^2)\mathrm{d}\tilde Y^2}{c^2(1-\tilde X^2-\tilde Y^2)}\right)\end{aligned} \tag{15.12}$$

which is clearly regular when $\tilde T=\tilde Z=0$.

In order to retain the one-to-one correspondence, it is appropriate initially to restrict the Bell–Szekeres coordinate x to the range $-\pi < cx \leq \pi$. The coordinates $\tilde X$ and $\tilde Y$ are restricted by $\tilde X^2+\tilde Y^2<1$, where the boundary $\tilde X^2+\tilde Y^2=1$ contains the space-like plane on which the collision occurs.

It has been shown by Clarke and Hayward (1989) that region IV is part of a space-time that is regular apart from two covering-space singularities that occur when $au-bv=\pm\pi/2$. These are points on the boundary of region IV where it joins with the limits of regions II and III. At these points, which are on the polar axis $\theta=0$ and $\theta=\pi$ in the Bertotti–Robinson form of the metric (15.9), $\tilde X=\tilde Y=0$.

With the above transformation, each interval of the coordinate x of length $2\pi/c$ corresponds to a circular region in the $\tilde X,\tilde Y$ plane with the origin removed and cut along the line $\tilde Y=0$, $\tilde X<0$. Neighbouring regions of x are joined together along these cuts to form a surface which continuously winds around the origin. The origin of the $\tilde X,\tilde Y$ plane thus corresponds to a singularity of a quasiregular covering-space type.

It is also sometimes convenient (Clarke and Hayward, 1989) to make the further coordinate transformation

$$\begin{aligned}\rho &= \sinh^{-1}\tilde{Z}\\ \chi &= \sin^{-1}\left(\frac{\tilde{T}}{\sqrt{1+\tilde{Z}^2}}\right)\\ \tilde{\theta} &= \cos^{-1}\tilde{Y}\\ \tilde{\phi} &= \sin^{-1}\left(\frac{\tilde{X}}{\sqrt{1-\tilde{Y}^2}}\right).\end{aligned}\tag{15.13}$$

With this, the line element (15.12) takes the alternative form of the Bertotti–Robinson line element

$$\mathrm{d}s^2 = \frac{1}{c^2}\left(\cosh^2\rho\,\mathrm{d}\chi^2 - \mathrm{d}\rho^2 - \mathrm{d}\tilde{\theta}^2 - \sin^2\tilde{\theta}\,\mathrm{d}\tilde{\phi}^2\right).\tag{15.14}$$

Region IV is now described by the line element (15.14) with the coordinate ranges given by

$$\begin{gathered}-\infty < \tilde{\phi} < \infty\\ -\infty < \rho < \infty\\ -\pi/2 < \chi < -\sin^{-1}|\tanh\rho|\\ \cos^{-1}(\cos\chi\cosh\rho) < \tilde{\theta} < \cos^{-1}(-\cos\chi\cosh\rho).\end{gathered}\tag{15.15}$$

This is joined to regions II and III across the surfaces

$$\cos\tilde{\theta} = \cos\chi\cosh\rho \qquad \text{and} \qquad \cos\tilde{\theta} = -\cos\chi\cosh\rho.\tag{15.16}$$

Clarke and Hayward (1989) have also shown that the Bell–Szekeres solution is extendable to the future through the surface $au + bv = \pi/2$, but the extension is not unique because of the presence of the singularities. They have further suggested two possible extensions which will be described in the next section.

It is also somewhat remarkable that the global structure of the Bell–Szekeres solution, as described by Clarke and Hayward (1989), is very similar to that of the collision of an impulsive gravitational wave with a null shell of matter, as described by Babala (1987). This will be discussed in Section 19.3.

Chandrasekhar and Xanthopoulos (1988) have conducted a thorough perturbation analysis of this particular solution. Throughout the interaction region subsequent to the collision they have obtained a complete set of bounded normal modes that are expressed in terms of spin-weighted spherical harmonics. These modes exhibit a behaviour of ever-increasing frequency as the Cauchy horizon is approached.

Chandrasekhar and Xanthopoulos (1988) have also considered perturbations in the initial regions which contain the approaching waves. For the approaching wave in region II which propagates along the null line $u=\text{const}$, $v\leq 0$, they have shown that v-independent perturbations are not permitted to all orders, while all the v-dependent perturbations exhibit strong divergences along the 'fold' singularity. Unfortunately, these perturbations can not be joined continuously to those of the interaction region. It is therefore not possible to consider the effects of these perturbations subsequent to the collision.

15.3 Extensions of the solution

In the Bell–Szekeres solution, it has been shown that the hypersurface $f+g=0$ on which the opposing waves mutually focus each other is a Cauchy horizon rather than a curvature singularity. It is therefore appropriate to consider possible extensions to this solution through the horizon, even though any such extension will not be unique. Two possible extensions have been given by Clarke and Hayward (1989).

One natural extension of region IV is simply the future extension of the region as described by the line element (15.14). This is achieved by extending the coordinate range to

$$\begin{gathered} -\infty<\tilde\phi,\rho<\infty, \qquad -\pi/2<\chi<\infty \\ \cos^{-1}\lambda<\tilde\theta<\cos^{-1}(-\lambda) \end{gathered} \tag{15.17}$$

where $\lambda=\min(1,\cos\chi\cos\rho)$. In this extension, the waves cross and focus. The subsequent space-time is unchanging.

Another equally natural extension of region IV is obtained by noting that the line element (15.14) admits a reflection symmetry $\chi\to-\chi$ about the hyperplane $\chi=0$ (or $\tilde T\to-\tilde T$ about $\tilde T=0$). Thus, there is an extension in which region IV is extended as far as $\chi=0$ and is then followed by the time reverse of the entire solution. In this extension, the waves cross, mutually focus each other, re-expand and then separate leaving Minkowski space between them to the future. This situation is illustrated in Figure 15.1 in terms of the original null coordinates and the line elements (15.3) and (15.7).

Other apparent extensions of the Bell–Szekeres solution have been suggested by Gürses and Halilsoy (1982) in which the approaching electromagnetic waves consist of a sequence of steps. The step lengths, however, have to be double the distance to the fold singularities. Thus, these

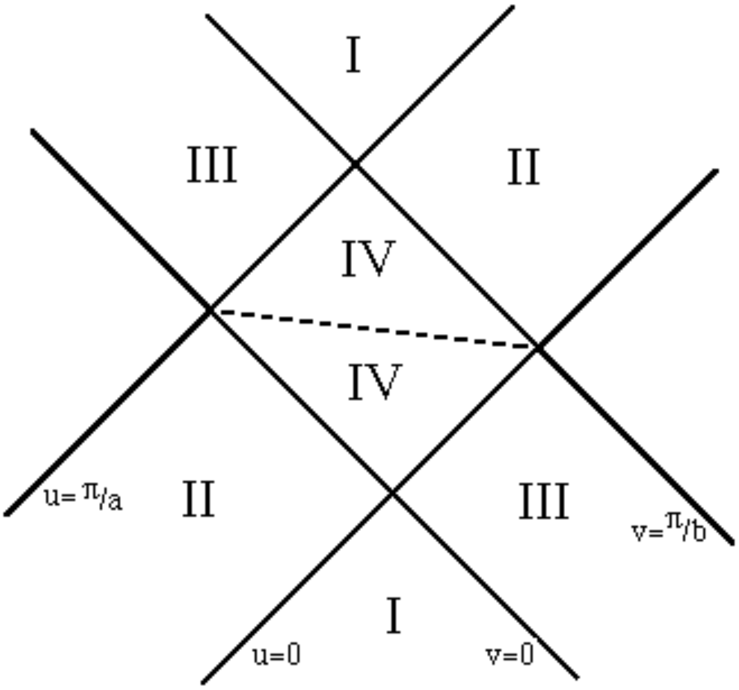

Figure 15.1 An extension of the Bell–Szekeres space-time. Regions marked I are flat. Regions marked II and III contain electromagnetic waves with $\Phi_2 = a$, and $\Phi_0 = b$ respectively, having metrics given by (15.3). There are impulsive gravitational waves along the boundaries of region IV, which has metric (15.7).

solutions necessarily extend the space-time through the covering space singularities and can not, therefore, be considered as appropriate solutions for colliding plane waves. In all these solutions the space-time can only be extended uniquely up to the fold singularities in regions II and III and the surface in region IV on which $au + bv = \pi/2$. It must therefore be concluded that the Gürses–Halilsoy extensions are not physically significant.

15.4 A non-colinear collision

In the Bell–Szekeres solution, the polarization of the approaching electromagnetic waves is aligned. It is appropriate now to consider how this solution may be generalized to include the case when the polarization is not aligned.

In the approaching waves a rotation of the polarization is expressed by a rotation of the components Φ_2 or Φ_0 in the complex plane. The metrics in both regions are unaltered, since a null electromagnetic field is defined by the metric only up to a constant duality rotation. Thus, to extend the Bell–Szekeres solution, the same initial conditions are given except that the component Φ_{02} becomes complex and, from (6.22e), W must become non-zero in the interaction region.

A solution presented by Griffiths (1985) satisfying this condition may be written in the form

$$\begin{aligned} e^{-U} &= \tfrac{1}{2}\cos 2au + \tfrac{1}{2}\cos 2bv \\ e^{2V} &= \frac{(1+\cos 2au\cos 2bv + \sin 2au\sin 2bv\cos\alpha)}{(1+\cos 2au\cos 2bv - \sin 2au\sin 2bv\cos\alpha)} \\ \sinh W &= \frac{\sin 2au\sin 2bv\sin\alpha}{(\cos 2au + \cos 2bv)} \\ M &= 0, \qquad \Phi_2 = ae^{i\theta}, \qquad \Phi_0 = ae^{-i\theta} \end{aligned} \tag{15.18}$$

where

$$\theta = \tfrac{1}{2}\cos^{-1}(\cos\alpha\,\mathrm{sech}W). \tag{15.19}$$

The relative polarization angle of the two waves prior to the collision is given by α. It may be noticed that the expression for U here is equivalent to those of (15.4,5) and (15.6). The common multiple 2 has been inserted so that this solution can be more directly related to the Bell–Szekeres solution to which it reduces when $\alpha = 0$.

It can be shown that this solution has exactly the same properties as the aligned Bell–Szekeres solution. It is conformally flat everywhere except at the boundaries of the interaction region where there are impulsive gravitational waves that may be considered to be generated by the collision. These are given by

$$\begin{aligned} \Psi_4 &= -a\tan bv\, e^{i\alpha}\delta(u)\Theta(v) \\ \Psi_0 &= -b\tan au\, e^{-i\alpha}\delta(v)\Theta(u). \end{aligned} \tag{15.20}$$

According to a theorem of Tariq and Tupper (1974), this new solution must also be the Bertotti–Robinson space-time in a different coordinate system. In fact it can be obtained from the Bell–Szekeres line element (15.7) by the simple rotation

$$\begin{aligned} x &= \cos\frac{\alpha}{2}x' + \sin\frac{\alpha}{2}y' \\ y &= -\sin\frac{\alpha}{2}x' + \cos\frac{\alpha}{2}y'. \end{aligned} \tag{15.21}$$

Thus, the gravitational field in this case is identical to that of the Bell–Szekeres solution, as may have been expected. In the collision of plane electromagnetic waves with non-aligned polarization, it must therefore be concluded that the dynamics remains unaltered. It is only the form of the metric in the interaction region that needs to be modified to that given in this section.